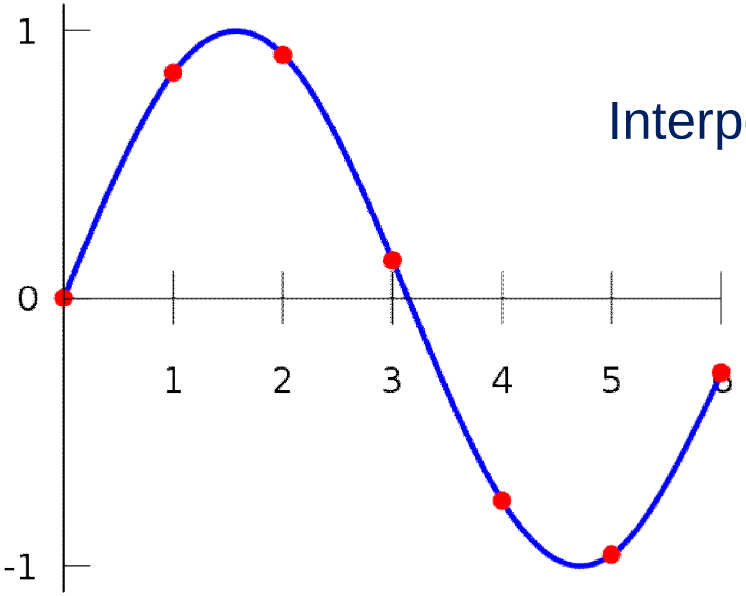
METODE NUMERIK

INTERPOLASI

Interpolasi Beda Terbagi Newton
Interpolasi Lagrange
Interpolasi Spline



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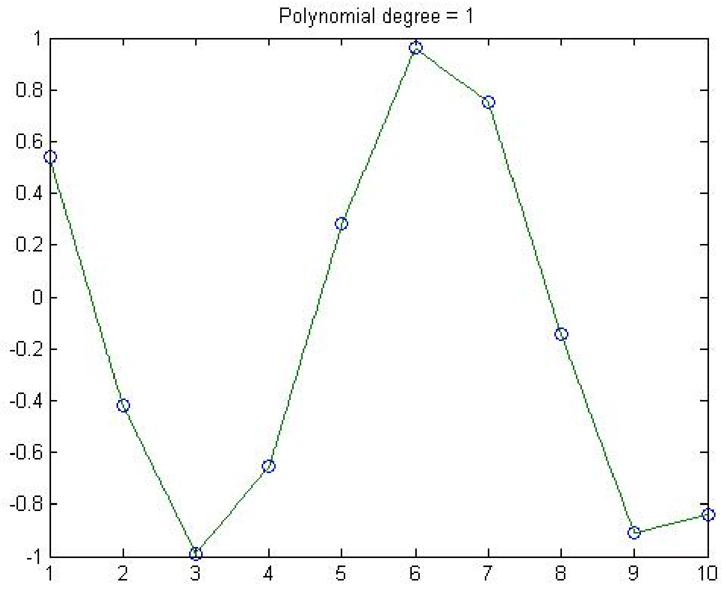
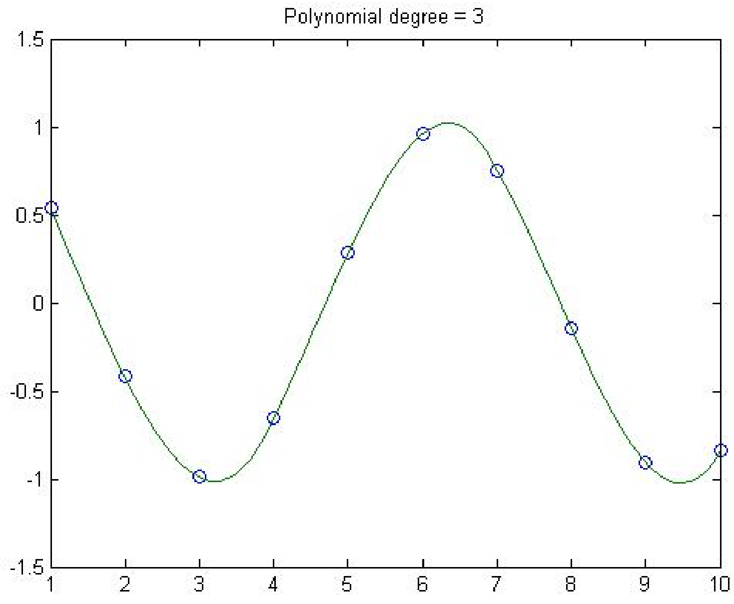
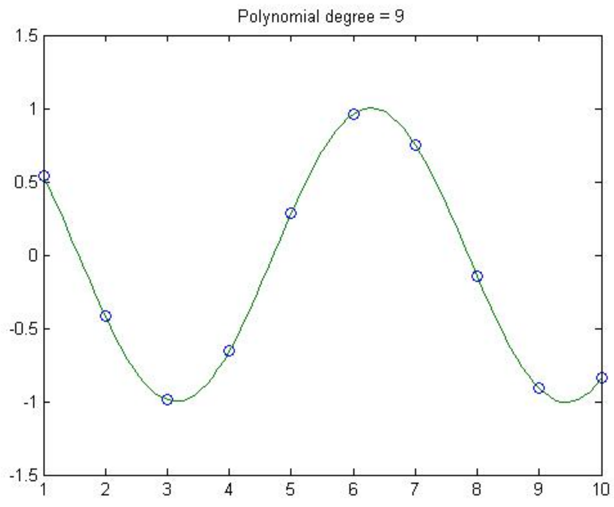
Interpolasi n-derajat polinom







Tujuan

- Interpolasi berguna untuk menaksir harga-harga tengah antara titik data yang sudah tepat. Interpolasi mempunyai orde atau derajat.

Macam Interpolasi Beda Terbagi Newton

- Interpolasi Linier

  Derajat/orde 1 $\rightarrow$ memerlukan 2 titik

x	f(x)
1	4,5
2	7.6
3	9.8
4	11.2

Berapa f(x = 1,325) = ?

Memerlukan 2 titik awal :

x = 1

x = 2

Macam Interpolasi Beda Terbagi Newton

- Interpolasi Kuadratik

  Derajat/orde 2 $\rightarrow$ memerlukan 3 titik

$$\left.\begin{array}{l} x = 1 \rightarrow f(x = 1) = \,.\,.\,.\,.\,. \\ x = 2 \rightarrow f(x = 2) = \,.\,.\,.\,.\,. \\ x = 3 \rightarrow f(x = 3) = \,.\,.\,.\,.\,. \end{array}\right\} \quad f\,(x = 1{,}325) = ?$$

Macam Interpolasi Beda Terbagi Newton

- Interpolasi Kubik

  Derajat/orde 3 $\rightarrow$ memerlukan 4 titik

…

- Interpolasi derajat/orde ke-n

  $\rightarrow$ memerlukan n+1 titik

- Semakin tinggi orde yang digunakan untuk interpolasi hasilnya akan semakin baik (teliti).

Interpolasi Linier

- Cara: menghubungkan 2 titik dengan sebuah garis lurus
- Pendekatan formulasi interpolasi linier sama dengan persamaan garis lurus.

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x - x_0)$$

Interpolasi Linier

- Prosentase kesalahan pola interpolasi linier :

$$\varepsilon_t = \frac{\text{Harga\_hasil\_perhitungan} - \text{Harga\_sebenarnya}}{\text{Harga\_sebenarnya}}$$

Interpolasi Linier (Ex.1)

- Diketahui suatu nilai tabel distribusi ‘Student t’ sebagai berikut :

  $t_5\% = 2{,}015$

  $t_{2,5}\% = 2{,}571$

  Berapa $t_4\% = ?$

Interpolasi Linier (Ex.1)

- Penyelesaian

  $x_0 = 5 \rightarrow f(x_0) = 2{,}015$

  $x_1 = 2{,}5 \rightarrow f(x_1) = 2{,}571$

  $x = 4 \rightarrow f(x) = ?$

  Dilakukan pendekatan dengan orde 1 :

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x - x_0)$$

$$= 2{,}015 + \frac{(2{,}571 - 2{,}015)}{2{,}5 - 5}(4 - 5)$$

$$= 2{,}2374 \approx 2{,}237$$

Interpolasi Linier (Ex.2)

- Diketahui:

  log 3 = 0,4771213

  log 5 = 0,698700
- Harga sebenarnya:

  log (4,5) = 0,6532125 (kalkulator).
- Harga yang dihitung dengan interpolasi:

  log (4,5) = 0,6435078

$$\varepsilon_t = \left| \frac{0{,}6435078 - 0{,}6532125}{0{,}6532125} * 100\% \right| = 1{,}49\%$$

Interpolasi Linier

- Pendekatan interpolasi dengan derajat 1, pada kenyataannya sama dengan mendekati suatu harga tertentu melalui garis lurus.
- Untuk memperbaiki kondisi tersebut dilakukan sebuah interpolasi dengan membuat garis yang menghubungkan titik yaitu melalui orde 2, orde 3, orde 4, dst, yang sering juga disebut interpolasi kuadratik, kubik, dst.

Interpolasi Kuadratik

- Interpolasi orde 2 sering disebut sebagai interpolasi kuadratik, memerlukan 3 titik data.
- Bentuk polinomial orde ini adalah :

$$f_2(x) = a_0 + a_1x + a_2x^2$$

dengan mengambil:

$$a_0 = b_0 – b_1x_0 + b_2x_0x_1$$

$$a_1 = b_1 – b_2x_0 + b_2x_1$$

$$a_2 = b_2$$

Interpolasi Kuadratik

- Sehingga

$$f_2(x) = \underbrace{b_0 + b_1(x-x_0)}_{\text{Pendekatan dengan garis linier}} + \underbrace{b_2(x-x_0)(x-x_1)}_{\text{Pendekatan dengan kelengkungan}}$$

dengan

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \rightarrow f[x_1, x_0]$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)} \rightarrow f[x_2, x_1, x_0]$$

Interpolasi Kubik

- $f_3(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2)$

dengan:

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \rightarrow f[x_1, x_0]$$

$$b_2 = \frac{f[x_2, x_1] - f[x_1, x_0]}{(x_2 - x_0)} = \frac{\dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)} \rightarrow f[x_2, x_1, x_0]$$

$$b_3 = \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{(x_3 - x_0)} \rightarrow f[x_3, x_2, x_1, x_0]$$

Interpolasi Beda Terbagi Newton

- Secara umum:

$f_1(x) = b_0 + b_1(x-x_0)$

$f_2(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1)$

$f_3(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) +$
$\qquad b_3(x-x_0)(x-x_1)(x-x_2)$

…

$f_n(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) +$
$\qquad b_3(x-x_0)(x-x_1)(x-x_2) + \ldots +$
$\qquad b_n(x-x_1)(x-x_2)\ldots(x-x_{n-1})$

Interpolasi Beda Terbagi Newton

Dengan:

- $b_0 = f(x_0)$
- $b_1 = f[x_1, x_0]$
- $b_2 = f[x_2, x_1, x_0]$

…

- $b_n = f[x_n, x_{n-1}, x_{n-2}, \; . \; . \; . \; ., x_0]$

Interpolasi Beda Terbagi Newton (Ex.)

- Hitung nilai tabel distribusi ‘Student t’ pada derajat bebas dengan α = 4%, jika diketahui:

  $t_{10\%} = 1,476$ $t_{2,5\%} = 2,571$

  $t_{5\%} = 2,015$ $t_{1\%} = 3,365$

  dengan interpolasi Newton orde 2 dan orde 3!

Interpolasi Beda Terbagi Newton (Ex.)

Interpolasi Newton Orde 2: → butuh 3 titik

- $x_0 = 5$ $\quad f(x_0) = 2{,}015$
  $x_1 = 2{,}5$ $\quad f(x_1) = 2{,}571$
  $x_2 = 1$ $\quad f(x_2) = 3{,}365$
- $b_0 = f(x_0) = 2{,}015$

$$b_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \frac{2{,}571 - 2{,}015}{2{,}5 - 5} = -0{,}222$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)}$$

$$= \frac{\dfrac{3{,}365 - 2{,}571}{1 - 2{,}5} - \dfrac{2{,}571 - 2{,}015}{2{,}5 - 5}}{1 - 5} = 0{,}077$$

Interpolasi Beda Terbagi Newton (Ex.)

- $f_2(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1)$

  $= 2{,}015 + (-0{,}222)\ (4-5) +$

  $\quad 0{,}077\ (4-5)(4-2{,}5)$

  $= 2{,}121$

Interpolasi Beda Terbagi Newton (Ex.)

Interpolasi Newton Orde 3: → butuh 4 titik

- $x_0 = 5$ $f(x_0) = 2{,}015$

  $x_1 = 2{,}5$ $f(x_1) = 2{,}571$

  $x_2 = 1$ $f(x_2) = 3{,}365$

  $x_3 = 10$ $f(x3) = 1{,}476$

Interpolasi Beda Terbagi Newton (Ex.)

- $b_0 = f(x_0) = 2{,}015$
  $b_1 = -0{,}222 \rightarrow f[x_1,x_0]$
  $b_2 = 0{,}077 \rightarrow f[x_2,x_1,x_0]$

$$b_3 = \frac{\dfrac{\dfrac{1{,}476 - 3{,}365}{10 - 1} - \dfrac{3{,}365 - 2{,}571}{1 - 2{,}5}}{10 - 2{,}5} - 0{,}077}{10 - 5}$$

$$= \frac{0{,}043 - 0{,}077}{5}$$

$$= -0{,}007$$

Interpolasi Beda Terbagi Newton (Ex.)

- $f_3(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2)$

  $= 2{,}015 + (-0{,}222)(4-5) + 0{,}077\,(4-5)(4-2{,}5) + (-0{,}007)(4-5)(4-2{,}5)(4-1)$

  $= 2{,}015 + 0{,}222 + 0{,}1155 + 0{,}0315$

  $= 2{,}153$

Kesalahan Interpolasi Beda Terbagi Newton

- $R_n = |f[x_{n+1},x_n,x_{n-1},\ldots,x_0](x-x_0)(x-x_1)\ldots(x-x_n)|$
- Menghitung R_1

  Perlu 3 titik (karena ada x_{n+1})

  $R_1 = |f[x_2,x_1,x_0](x-x_0)(x-x_1)|$
- Menghitung R_2

  Perlu 4 titik sebagai harga awal

  $R_2 = |f[x_3,x_2,x_1,x_0](x-x_0)(x-x_1)(x-x_2)|$

Kesalahan Interpolasi Beda Terbagi Newton (Ex.)

- Berdasarkan contoh:

$R_1 = |f[x_2,x_1,x_0](x-x_0)(x-x_1)|$

$= |0.077\ (4-5)(4-2.5)|$

$= 0.1155$

$R_2 = |f[x_3,x_2,x_1,x_0](x-x_0)(x-x_1)(x-x_2)|$

$= |-0.007\ (4-5)(4-2.5)(4-1)|$

$= 0.0315$

Interpolasi Lagrange

- Interpolasi Lagrange pada dasarnya dilakukan untuk menghindari perhitungan dari differensiasi terbagi hingga (Interpolasi Newton)
- Rumus: $$f_n(x) = \sum_{i=0}^{n} L_i(x).f(x_i)$$

dengan $$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Interpolasi Lagrange

- Pendekatan orde ke-1

  $f_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \qquad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$\therefore f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

Interpolasi Lagrange

- Pendekatan orde ke-2

$f_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$

$$\underset{\substack{i=0\\n=2\\j\neq i}}{L_0}(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right) \qquad \underset{\substack{i=1\\n=2\\j\neq i}}{L_1}(x) = \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)$$

$$\underset{\substack{i=2\\n=2\\j\neq i}}{L_2}(x) = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)$$

$$\therefore\ f_2(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)f(x_1) + \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)f(x_2)$$

Interpolasi Lagrange

- Pendekatan orde ke-3

$$f_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)$$

$$f_2(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)\left(\frac{x - x_3}{x_0 - x_3}\right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right) f(x_1) +$$

$$\left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right) f(x_2) + \left(\frac{x - x_0}{x_3 - x_0}\right)\left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right) f(x_3)$$

Interpolasi Lagrange (Ex.)

- Berapa nilai distribusi t pada α = 4 %?

$\alpha = 2{,}5\ \%\ \rightarrow x_0 = 2{,}5 \rightarrow f(x_0) = 2{,}571$

$\alpha = 5\ \%\ \rightarrow x_1 = 5 \rightarrow f(x_1) = 2{,}015$

$\alpha = 10\ \%\ \rightarrow x_2 = 10 \rightarrow f(x_2) = 1{,}476$

Interpolasi Lagrange (Ex.)

- Pendekatan orde ke-1

$f_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$= \left(\frac{4 - 5}{2,5 - 5}\right)(2,571) + \left(\frac{4 - 2,5}{5 - 2,5}\right)(2,015)$$

$$= \underline{\underline{2,237}}$$

Interpolasi Lagrange (Ex.)

- Pendekatan orde ke-2

  $f_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$

$$\therefore f_2(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right) f(x_1) + \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right) f(x_2)$$

$$= \left(\frac{4 - 5}{2{,}5 - 5}\right)\left(\frac{4 - 10}{2{,}5 - 10}\right)(2{,}571) + \left(\frac{4 - 2{,}5}{5 - 2{,}5}\right)\left(\frac{4 - 10}{5 - 10}\right)(2{,}015) + \left(\frac{4 - 2{,}5}{10 - 2{,}5}\right)\left(\frac{4 - 5}{10 - 5}\right)(1{,}476)$$

$$= \underline{\underline{2{,}214}}$$

Interpolasi Spline

- Tujuan: penghalusan
- Interpolasi spline linear, kuadratik, kubik.

Interpolasi Cubic Spline

$$S(x) = \begin{cases} s_1(x) & \text{if} \quad x_1 \le x < x_2 \\ s_2(x) & \text{if} \quad x_2 \le x < x_3 \\ & \vdots \\ s_{n-1}(x) & \text{if} \quad x_{n-1} \le x < x_n \end{cases}$$

dimana S_i adalah polinomial berderajat 3:

$p(x_i) = d_i + (x-x_i)\, c_i + (x-x_i)^2\, b_i + (x-x_i)^3\, a_i$, i=1,2, …, n-1

Syarat: $S_i(x_i) = S_{i+1}(x_i)$, $S_i'(x_i) = S_{i+1}'(x_i)$, $S_i''(x_i) = S_{i+1}''(x_i)$

Interpolasi Cubic Spline

- Interpolasi spline kubik menggunakan polinomial p(x) orde 3

  $p(x) = d_i + (x-x_i)\, c_i + (x-x_i)^2\, b_i + (x-x_i)^3\, a_i$

- Turunan pertama dan kedua $p(x_i)$ yaitu:

  $p'(x) = c_i + 2b_i\,(x-x_i) + 3a_i\,(x-x_i)^2$

  $p''(x) = 2b_i + 6a_i\,(x-x_i)$

Interpolasi Cubic Spline

- Evaluasi pada titik $x=x_i$ menghasilkan:

  $p_i = p(x_i) = d_i$

  $p_i'' = p''(x_i) = 2b_i$

- Evaluasi pada titik $x=x_{i+1}$ menghasilkan:

  $p_i \quad = d_i + (x_{i+1}-x_i)\, c_i + (x_{i+1}-x_i)^2\, b_i + (x_{i+1}-x_i)^3\, a_i$

  $p(x_i) \quad = d_i + h_i\, c_i + h_i^2\, b_i + h_i^3\, a_i$

  $p''_i \quad = 2b_i + 6a_i\, (x_{I+1}-x_i)$

  $p''(x_{i+1}) \quad = 2b_i + 6a_i\, h_i$

  dimana $h_i = (x_{I+1}-x_i)$

Interpolasi Cubic Spline

- Jadi:

$d_i = p_i$

$$b_i = \frac{p_i''}{2}$$

$$a_i = \frac{p''_{i+1} - p''_i}{6h_i} \qquad c_i = \frac{p_{i+1} - p_i}{h_i} - \frac{h_i p''_{i+1} + 2h_i p''_i}{6}$$

- Sehingga:

$$p(x) = p_i + \left(\frac{p_{i+1} - p_i}{h_i} - \frac{h_i p''_{i+1}}{6} - \frac{h_i p''_i}{3}\right)(x - x_i) + \frac{p''_i}{2}(x - x_i)^2 + \left(\frac{p''_{i+1} - p''_i}{6h_i}\right)(x - x_i)^3$$