Tranformasi Z
[Sistem Kontrol Digital]

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Mengapa diperlukan Transformasi-Z?

Sinyal pada domain Diskrit Time

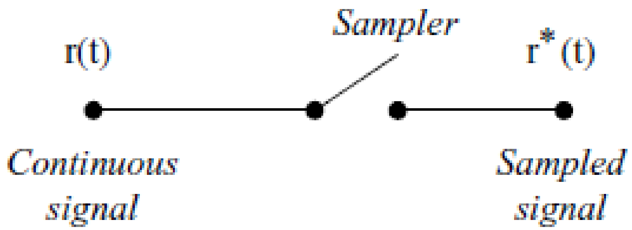
Ragazzini & Zadeh
pada
kontrol
data tersampling

Sinyal pada Domain Frekuensi

Output dari respon sistem data tersampling dapat ditentukan dengan mudah

Definisi

$$R^*(s) = \sum_{n=0}^{\infty} r(nT)e^{-snT}$$



$$e^{-snT}$$

$$Z = e^{sT}$$

Definisi Transformasi Z

Transformasi Z dari *r(t)* $\Rightarrow$ $Z[r(t)] = R(z)$

Variabel Infinite series Transformasi-Z

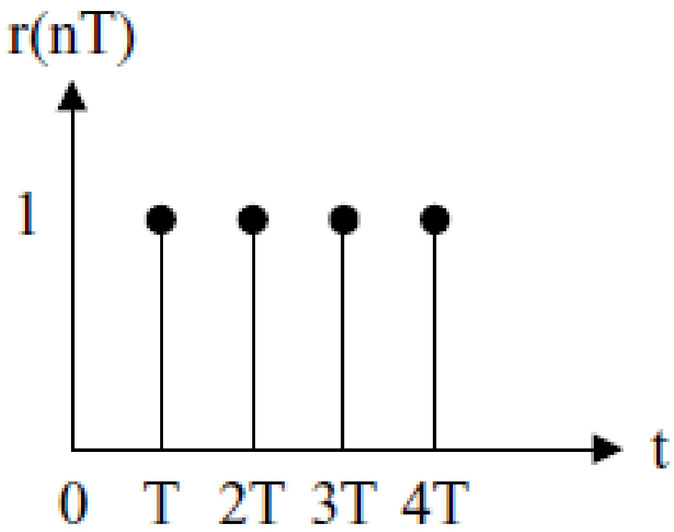
Transformasi Z dari *r(t)*

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n}$$

n=0,1,2,3,....

$$R(z) = r(0) + r(T)z^{-1} + r(2T)z^{-2} + r(3T)z^{-3} + \ldots$$

Fungsi Unit Step

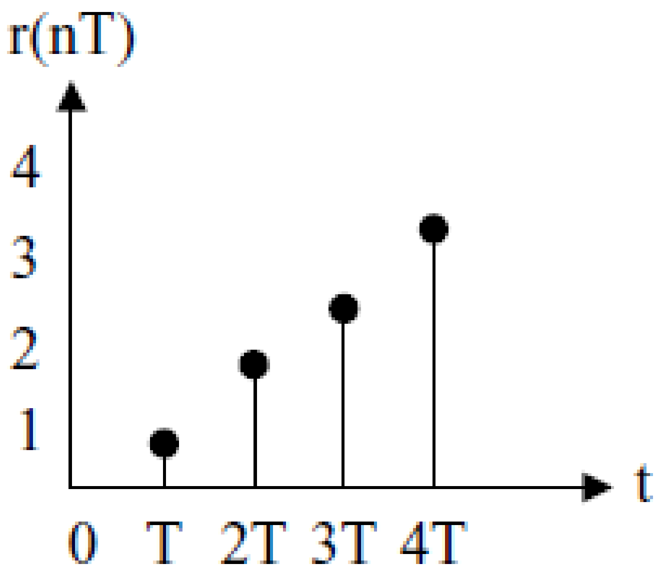


$$r(nT) = \begin{cases} 0, & n < 0, \\ 1, & n \geq 0. \end{cases}$$

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \ldots$$

$$R(z) = \frac{z}{z-1}, \quad \text{for } |z| > 1$$

Fungsi Unit Ramp

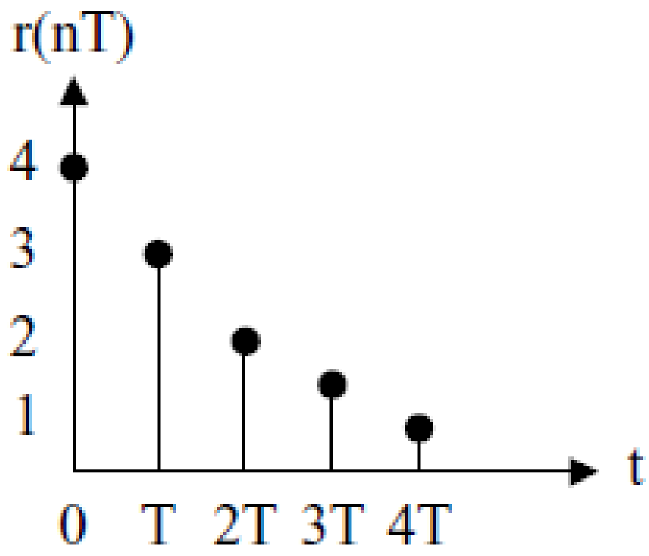


$$r(nT) = \begin{cases} 0, & n < 0, \\ nT, & n \geq 0. \end{cases}$$

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n} = \sum_{n=0}^{\infty} nT z^{-n} = Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + 4Tz^{-4} + \ldots$$

$$R(z) = \frac{Tz}{(z-1)^2}, \qquad \text{for } |z| > 1.$$

Fungsi Eksponensial



$$r(nT) = \begin{cases} 0, & n < 0, \\ e^{-anT}, & n \geq 0. \end{cases}$$

$$R(z) = \sum_{n=0}^{\infty} e^{-anT} z^{-n} = 1 + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + e^{-3aT} z^{-3} + \ldots$$

$$R(z) = \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}}, \quad \text{for } |z| < e^{-aT}$$

Fungsi Eksponensial Umum

$$r(n) = \begin{cases} 0, & n < 0, \\ p^n, & n \geq 0. \end{cases}$$

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n} = \sum_{n=0}^{\infty} p^n z^{-n} = 1 + pz^{-1} + p^2 z^{-2} + p^3 z^{-3} + \ldots$$

$$R(z) = \frac{z}{z-p}, \quad \text{for } |z| < |p|$$

$$R(p^{-k}) = \frac{z}{z - p^{-1}}$$

Fungsi Sinus

$$r(nT) = \begin{cases} 0, & n < 0, \\ \sin n\omega T. & n \geq 0. \end{cases}$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$r(nT) = \frac{e^{jn\omega T} - e^{-jn\omega T}}{2j} = \frac{e^{jn\omega T}}{2j} - \frac{e^{-jn\omega T}}{2j}$$

$$R(z) = \frac{1}{2j}\left(\frac{z}{z - e^{j\omega T}} - \frac{z}{z - e^{-j\omega T}}\right) = \frac{1}{2j}\left(\frac{z(e^{j\omega T} - e^{-j\omega T})}{z^2 - z(e^{j\omega T} + e^{-j\omega T}) + 1}\right)$$

$$R(z) = \frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$$

Fungsi Cosinus

$$r(nT) = \begin{cases} 0, & n < 0, \\ \cos n\omega T, & n \geq 0. \end{cases} \qquad \cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$r(nT) = \frac{e^{jn\omega T} + e^{-jn\omega T}}{2} = \frac{e^{jn\omega T}}{2} + \frac{e^{-jn\omega T}}{2}$$

$$R(z) = \frac{1}{2}\left(\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right)$$

$$R(z) = \frac{z(z - \cos\omega T)}{z^2 - 2z\cos\omega T + 1}$$

Fungsi Impulse Diskrit

$$\delta(n) = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1$$

Fungsi Impulse Diskrit Tunda

$$\delta(n-k) = \begin{cases} 1, & n = k > 0, \\ 0, & n \neq k. \end{cases}$$

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n} = \sum_{n=0}^{\infty} z^{-n} = z^{-n}$$

Tabel Transformasi Z

$f(kT)$	$F(z)$
$\delta(t)$	1
1	$\dfrac{z}{z-1}$
kT	$\dfrac{Tz}{(z-1)^2}$
e^{-akT}	$\dfrac{z}{z-e^{-aT}}$
kTe^{-akT}	$\dfrac{Tze^{-aT}}{(z-e^{-aT})^2}$
a^k	$\dfrac{z}{z-a}$
$1-e^{-akT}$	$\dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\sin akT$	$\dfrac{z\sin aT}{z^2-2z\cos aT+1}$
$\cos akT$	$\dfrac{z(z-\cos aT)}{z^2-2z\cos aT+1}$

Properti Transformasi Z

Linearitas

$$Z[f(nT) \pm g(nT)] = Z[f(nT)] \pm Z[g(nT)] = F(z) \pm G(z)$$

$$Z[af(nT)] = aZ[f(nT)] = aF(z)$$

Pergeseran ke Kiri $y(nT) = f(nT + mT)$

$$Y(z) = z^m F(z) - \sum_{i=0}^{m-1} f(iT) z^{m-i}$$

$f(iT) = 0, i = 0, 1, 2, \ldots, m-1$

$$Z[f(nT + mT)] = z^m F(z)$$

Properti Transformasi Z

Pergeseran ke Kanan $y(nT) = f(nT - mT)$

$$Y(z) = z^{-m}F(z) + \sum_{i=0}^{m-1} f(iT - mT)z^{-i}$$

$f(nT) = 0 \text{ for } k < 0,$

$$Z[f(nT - mT)] = z^{-m}F(z)$$

Attenuation

$$Z[e^{-anT}f(nT)] = F[ze^{aT}]$$

Properti Transformasi Z

Teorema Ninai Awal

$$\lim_{n\to 0} f(nT) = \lim_{z\to\infty} F(z)$$

Teorema Nilai Akhir

$$\lim_{n\to\infty} f(nT) = \lim_{z\to 1}(1 - z^{-1})F(z)$$

Contoh #1

Tentukan Transformasi Z dari 5r(nT) jika Transformasi Z dari r(T) adalah

$$R(z) = \frac{Tz}{(z-1)^2}$$

Solusi:

Dipecahkan dengan properti LINEARITAS

$$Z[5r(nT)] = 5R(z) = \frac{5Tz}{(z-1)^2}$$

Contoh #2

Jika Transformasi Z dari $r(nT) = \sin nwT$

$$R(z) = \frac{z \sin wT}{z^2 - 2z \cos wT + 1}$$

Tentukan transformasi Z dari $y(nT) = e^{-2T} \sin nWT$

Solusi:

Dipecahkan dengan properti Attenuasi

$$Z[y(nT)] = Z[e^{-2T} r(nT)] = R[ze^{2T}]$$

Contoh #2

Solusi: $y(nT) = e^{-2T} \sin nWT \quad R(z) = \dfrac{z \sin wT}{z^2 - 2z \cos wT + 1}$

Maka:

$$Z[y(nT)] = \frac{ze^{2T} \sin wT}{(ze^{2T})^2 - 2ze^{2T} \cos wT + 1}$$

$$= \frac{ze^{2T} \sin wT}{z^2 e^{4T} - 2ze^{2T} \cos wT + 1}$$

$$Z[y(nT)] = \frac{ze^{-2T} \sin wT}{z^2 - 2ze^{-2T} + e^{-4T}}$$

Latihan

Jika suatu fungsi adalah

$$G(z) = \frac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)}$$

Dengan teorema nilai akhir, tentukan g(nT)